Dynamic Stability of Viscoelastic Bars under Pulsating Axial Loads

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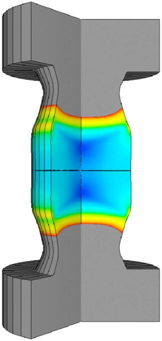
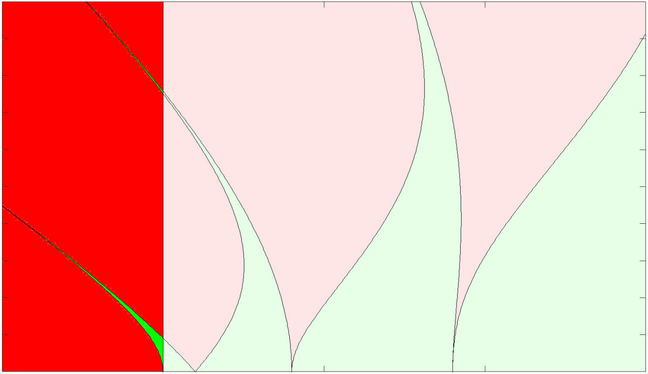
Motivation

Rather academic question, is it possible to go beyond the critical static load?



rubber (EPDM) specimen [1]



Ince-Strutt Diagram (Mathieu's equation) Inverted Pendulum [2]

[1] L. Kanzenbach, *Experimentell-numerische Vorgehensweise zur Entwicklung . . .*, Diss. TU Chemnitz, 2019

[2] https://sciencedemonstrations.fas.harvard.edu/presentations/inverted-pendulum

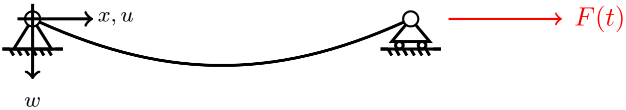
State of the Art

- Dynamic stability of elastic bars under pulsating loads is known since 1924 (N.M. Belaev), we follow along the lines of Weidenhammer [3].
- Rubber is a viscoelastic material and the Standard Linear Solid model, a.k.a. Zener model, is adequate for the general behavior, as it describes (roughly) both creep and stress relaxation [4].
- Compression tests for rubber are of high relevance, since compression is a frequent load case in applications and there is a tension-compression asymmetry, but these tests are prone to buckling.

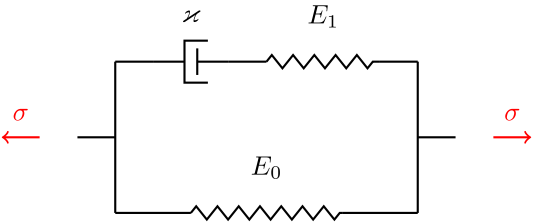
[3] F. Weidenhammer, *Nichtlineare Biegeschwingungen des axial-pulsierend . . .* , Ing.-Archiv 20.5, 1952

[4] I.M. Ward and J. Sweeney, *An Introduction to the Mechanical Properties . . .* , John Wiley & Sons, 2005

Geometry and Material



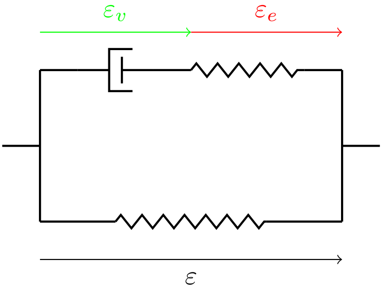
axially loaded beam (Euler buckling, case II)



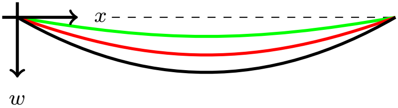
Standard Linear Solid model

Kinematics

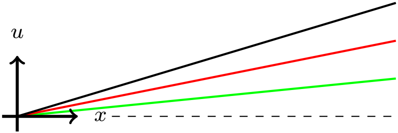
Euler-Bernoulli beam theory



$$\varepsilon = \varepsilon_v + \varepsilon_e$$



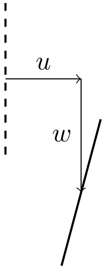
$$w(x,t) = w_v(x,t) + w_e(x,t)$$



$$u(x,t) = u_v(x,t) + u_e(x,t)$$

Hamilton's Principle

The dynamics are determined by the energy expressions



$$\mathcal{T} = \frac{1}{2}\int_0^l \varrho A(\dot{u}^2 + \dot{w}^2)\,\mathrm{d}x,$$

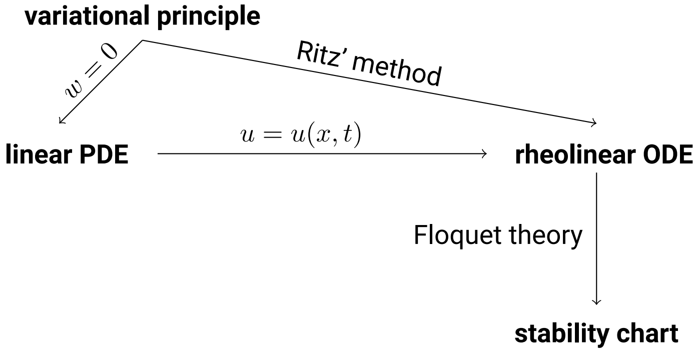
$$\mathcal{V} = \frac{1}{2}\int_0^l E_0 I w''^2 + E_0 A\left(u' + \frac{1}{2}w'^2\right)^2 \mathrm{d}x + \frac{1}{2}\int_0^l E_1 I w_e''^2 + E_1 A\left(u_e' + \frac{1}{2}w_e'^2\right)^2 \mathrm{d}x,$$

$$\delta\mathcal{W}^{\mathrm{nc}} = F\delta u(l) - \int_0^l \varkappa A \frac{\mathrm{d}}{\mathrm{d}t}\left(u_v' + \frac{1}{2}w_v'^2\right)\delta\left(u_v' + \frac{1}{2}w_v'^2\right) + \varkappa I \dot{w}_v''\delta w_v''\,\mathrm{d}x,$$

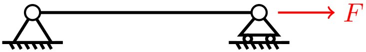
where the nonlinear strain measure $\mathbf{E} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\mathbf{H})$ has been evaluated for the potential energy.

Solution Strategy

Preliminary consideration: Axial loads excite only longitudinal vibrations in the stable regime, meaning $u = \mathcal{O}(1)$ while $w = \mathcal{O}(\varepsilon)$ with $\varepsilon \ll 1$.



Longitudinal Vibrations



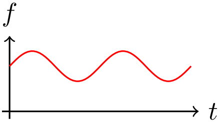
In absence of bending, Hamilton's principle leads to the linear PDE

$$\begin{aligned} \ddot{u} - (c_0^2 + c_1^2)u'' + c_1^2 u_v'' &= 0, \\ d\dot{u}_v'' - c_1^2 u'' + c_1^2 u_v'' &= 0, \end{aligned}$$

with $c_0^2 = \frac{E_0}{\varrho}$, $c_1^2 = \frac{E_1}{\varrho}$, $d = \frac{\varkappa}{\varrho}$, $f = \frac{F}{\varrho A}$ and the BC

$$\begin{aligned} 0 &= u(0,t), \\ 0 &= u_v(0,t), \\ f(t) &= (c_0^2 + c_1^2)u'(l,t) - c_1^2 u_v'(l,t), \\ 0 &= d\dot{u}_v'(l,t) - c_1^2 u'(l,t) + c_1^2 u_v'(l,t). \end{aligned}$$

Longitudinal Vibrations



For a pulsating load $f(t) = \bar{f} + \tilde{f} e^{\Lambda t}$ the solution reads

$$
\begin{aligned}
u(x,t) &= \frac{\bar{f}}{c_0^2} x + \frac{\tilde{f}}{Ck} \frac{e^{kx} - e^{-kx}}{e^{kl} + e^{-kl}} e^{\Lambda t}, \\
u_v(x,t) &= \frac{\bar{f}}{c_0^2} x + R \frac{\tilde{f}}{Ck} \frac{e^{kx} - e^{-kx}}{e^{kl} + e^{-kl}} e^{\Lambda t},
\end{aligned}
$$

with

$$
\begin{aligned}
\Lambda &= i\Omega, \\
R &= \frac{c_1^2}{c_1^2 + d\Lambda}, \\
C &= c_0^2 + c_1^2 (1 - R), \\
k &= \frac{\Lambda}{\sqrt{C}}.
\end{aligned}
$$

Induced Bending Vibrations

Ritz' method with

- prescribed longitudinal vibrations $u(x,t)$ and $u_v(x,t)$ excited by $f(t) = \bar{f} + \tilde{f}\,\mathfrak{Re}\{e^{i\Omega t}\} = \bar{f} + \tilde{f}\cos\Omega t$,
- one-term trial functions $w(x,t) = W(x)T(t)$ and $w_v(x,t) = W(x)T_v(t)$,

leads to a differential-algebraic equation (DAE) with $\mathbf{T} = \begin{bmatrix} T(t) \\ T_v(t) \end{bmatrix}$

$$\mathbf{M}\ddot{\mathbf{T}} + \mathbf{D}\dot{\mathbf{T}} + \mathbf{K}(t)\mathbf{T} = \mathbf{0}$$

and the matrices

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & d_1 \end{bmatrix}, \quad \mathbf{K}(t) = \begin{bmatrix} k_1(t) + k_2(t) & -k_2(t) \\ -k_2(t) & k_2(t) + k_3(t) \end{bmatrix}.$$

Induced Bending Vibrations

$$\begin{bmatrix} m_1 \ddot{T} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d_1 \dot{T}_v \end{bmatrix} + \begin{bmatrix} k_1(t) + k_2(t) & -k_2(t) \\ -k_2(t) & k_2(t) + k_3(t) \end{bmatrix} \begin{bmatrix} T \\ T_v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

with the coefficients

$$\begin{aligned} m_1 &= \int_0^l \varrho A W(x)^2 \, \mathrm{d}x, \\ d_1 &= \int_0^l \varkappa I W''(x)^2 \, \mathrm{d}x, \\ k_1(t) &= \int_0^l E_0 I \, W''(x)^2 + E_0 A \, \mathfrak{Re}\{u'(x,t)\} \, W'(x)^2 \, \mathrm{d}x, \\ k_2(t) &= \int_0^l E_1 I \, W''(x)^2 + E_1 A \, \mathfrak{Re}\{u_e'(x,t)\} \, W'(x)^2 \, \mathrm{d}x, \\ k_3(t) &= \int_0^l \varkappa A \, \mathfrak{Re}\{\dot{u}_v'(x,t)\} \, W'(x)^2 \, \mathrm{d}x. \end{aligned}$$

Induced Bending Vibrations

Remember, the longitudinal vibrations depend on the axial forcing

$$
\begin{aligned}
u'(x,t) &= \bar{F}\bar{U}'(x) + \tilde{F}\tilde{U}'(x)e^{i\Omega t}, \\
u_v'(x,t) &= \bar{F}\bar{U}'(x) + \tilde{F}R\tilde{U}'(x)e^{i\Omega t}, \\
u_e'(x,t) &= \tilde{F}\big(1-R\big)\tilde{U}'(x)e^{i\Omega t},
\end{aligned}
$$

with $R = \dfrac{E_1}{E_1 + \varkappa\Lambda}$ and the spatial coefficient functions

$$
\begin{aligned}
\bar{U}'(x) &= \frac{1}{E_0 A}, \\
\tilde{U}'(x) &= \frac{1}{E_0 A + (1-R)E_1 A}\,\frac{\cosh kx}{\cosh kl},
\end{aligned}
$$

corresponding to static and dynamic forcing, respectively.

Minimal Model

Nondimensionalization with length l, first angular eigenfrequency (bending) of the unloaded elastic bar ω_0, and static buckling load F_{crit} leads to

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} + \boldsymbol{\kappa} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where the nondimensional stiffness matrix reads

$$\boldsymbol{\kappa} = \begin{bmatrix} 1 + \delta + \alpha + \epsilon\Big(\tilde{\kappa}_1(\tau) + \tilde{\kappa}_2(\tau)\Big) & -\alpha - \epsilon\tilde{\kappa}_2(\tau) \\ \text{sym.} & \alpha + 2\epsilon\tilde{\kappa}_2(\tau) \end{bmatrix}.$$

Geometry parameter $\gamma = l\sqrt{\frac{A}{I}}$,

material parameters $\alpha = \frac{E_1}{E_0}$, $\beta = \frac{\varkappa\omega_0}{E_0}$,

excitation parameters $\delta = \frac{\bar{F}}{F_{\text{crit}}}$, $\epsilon = \frac{\tilde{F}}{F_{\text{crit}}}$, $\eta = \frac{\Omega}{\omega_0}$.

Minimal Model

The periodic coefficients of the nondimensional stiffness matrix $\boldsymbol{\kappa}$ read

$$\begin{aligned}
\tilde{\kappa}_1(\tau) &= \mathfrak{Re}\{\tilde{\kappa}\}\cos\eta\tau - \mathfrak{Im}\{\tilde{\kappa}\}\sin\eta\tau, \\
\tilde{\kappa}_2(\tau) &= \mathfrak{Re}\{\upsilon\tilde{\kappa}\}\cos\eta\tau - \mathfrak{Im}\{\upsilon\tilde{\kappa}\}\sin\eta\tau,
\end{aligned}$$

with

$$\begin{aligned}
\tilde{\kappa} &= 2\chi\frac{2\pi^2 + \Xi^2}{(4\pi^2 + \Xi^2)\Xi}\tanh\Xi, \\
\upsilon &= \frac{(\eta\beta + i\alpha)\alpha\eta\beta}{\alpha^2 + \eta^2\beta^2}, \\
\Xi &= \frac{i\pi^2\eta\sqrt{\chi}}{\gamma}, \\
\chi &= \frac{\alpha^2(1 - i\eta\beta) + \eta^2\beta^2(1+\alpha)}{\alpha^2 + \eta^2\beta^2(1+\alpha)^2}.
\end{aligned}$$

Floquet Theory

In state space we have an ordinary differential equation (ODE)

$$\begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \ddot{\varphi}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \dfrac{\alpha + \epsilon\tilde{\kappa}_2(\tau)}{\beta} & -\dfrac{\alpha + 2\epsilon\tilde{\kappa}_2(\tau)}{\beta} & 0 \\ -1 - \delta - \alpha - \epsilon\Big(\tilde{\kappa}_1(\tau) + \tilde{\kappa}_2(\tau)\Big) & \alpha + \epsilon\tilde{\kappa}_2(\tau) & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \dot{\varphi}_1 \end{bmatrix}.$$

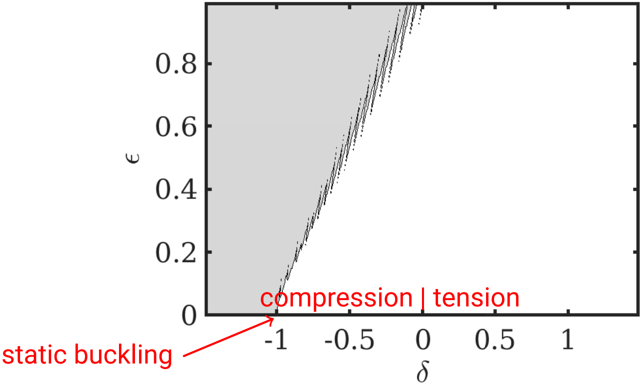
Numerical integration for one period $\tau = 0 \ldots \frac{2\pi}{\eta}$ with a set of unit initial conditions gives the monodromy matrix $\mathbf{P}$.

The eigenvalues λ_i of $\mathbf{P}$ determine stability (stable if all $|\lambda_i| < 1$) [5].

[5] H. Troger and A. Steindl, *Nonlinear Stability and Bifurcation Theory*, Springer, 1991

Stability Chart

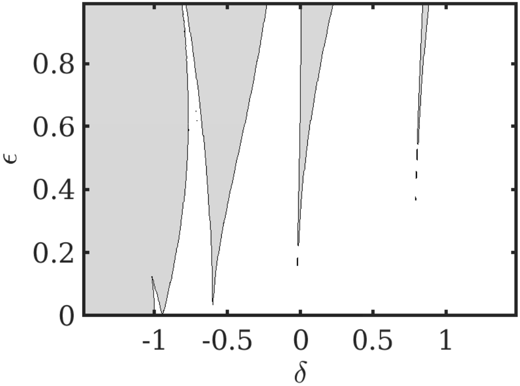
$\eta = 0.067$ $\eta = 0.67$ $\eta = 6.7$



stable regions (white) of the straight bar w.r.t. static and harmonic load

Stability Chart

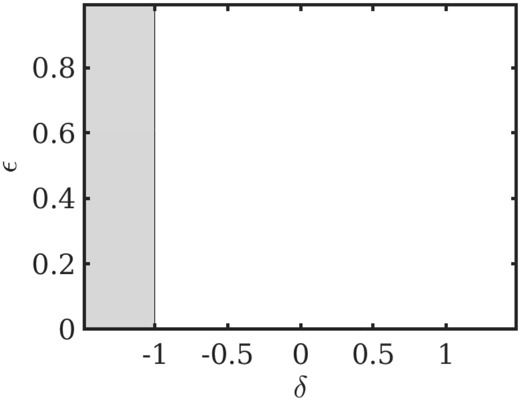
$\eta = 0.067$ $\eta = 0.67$ $\eta = 6.7$



stable regions (white) of the straight bar w.r.t. static and harmonic load

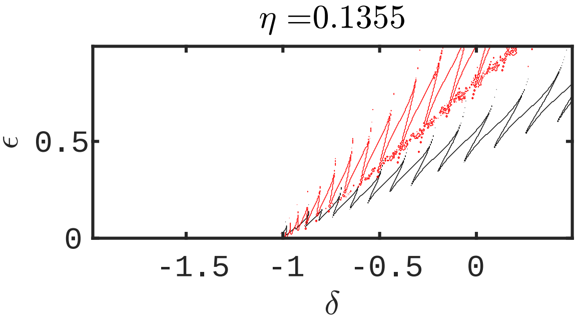
Stability Chart

$\eta = 0.067$ $\eta = 0.67$ $\eta = 6.7$



stable regions (white) of the straight bar w.r.t. static and harmonic load

Comparison



viscoelastic bar (black) and elastic bar (red)

Summary and Outlook

- Euler-Bernoulli beam theory, Standard Linear Solid (Zener model), one-sided coupling, Ritz' method, Floquet theory;
- dependency of stability on excitation parameters (static offset, amplitude, frequency) has been studied, it is theoretically possible to load the bar with more than twice the static buckling load;
- compared to the elastic bar, viscosity seems to reduce the size of stable regions.

- Resolve correspondence between discrete and continuous model;
- analyze dependency of stability on further parameters (material, geometry);
- study high excitation frequencies (multi-mode discretization) and different boundary conditions;
- work around the experimental problem (*real world*), that stable regime depends on parameters to be measured/updated.

Elastic Bar Statically Loaded

- buckling load (same for viscoelastic bar)

$$F_{\text{crit}} = -\frac{\pi^2}{l^2} E_0 I = -50.446\,\text{N}$$

- base radian eigenfrequency, longitudinal oscillations

$$\begin{aligned} \omega^2_{0,E_0} &= \frac{E_0}{\varrho}\left(\frac{\pi}{2l}\right)^2 \\ f_{0,E_0} &= 375.7\,\text{Hz} \qquad (\eta = 2.55) \end{aligned}$$

- base radian eigenfrequency, bending oscillations

$$\begin{aligned} \omega^2_{0,E_0} &= \frac{\pi^2}{l^2}\left(\frac{\pi^2}{l^2}\frac{E_0 I}{\varrho A} + \frac{\bar{F}}{\varrho A}\right) \\ f_{0,E_0}(\bar{F} = 0) &= 147.547\,\text{Hz} \qquad (\eta = 1) \end{aligned}$$

Viscoelastic Bar Statically Loaded

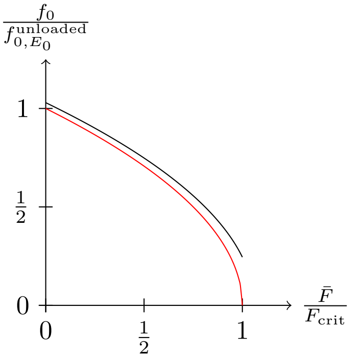
longitudinal oscillations

$$0 = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$
$$a_2 = \frac{E_1}{\varkappa}$$
$$a_1 = \frac{E_0 + E_1}{\varrho}\left(\frac{\pi}{2l}\right)^2$$
$$a_0 = \frac{E_0 E_1}{\varkappa\varrho}\left(\frac{\pi}{2l}\right)^2$$

oscillation ($\lambda_1 = \bar{\lambda}_2$) and decay ($\lambda_3$)

$$f_0 = \frac{|\mathfrak{Im}\{\lambda_{1,2}\}|}{2\pi} = 386.8\,\text{Hz}$$
$$\delta_0 = -\mathfrak{Re}\{\lambda_{1,2}\} = 0.18\,\text{s}^{-1}$$
$$\delta_1 = -\mathfrak{Re}\{\lambda_3\} = 6.01\,\text{s}^{-1}$$

bending oscillations



viscoelastic (Zener)
elastic (Hook)

Example Parameters

Geometry

$$
\begin{aligned}
r &= 0.007\,\mathrm{m} \\
l &= 0.056\,\mathrm{m} \\
\gamma &= 16.00
\end{aligned}
$$

Material

$$
\begin{aligned}
\varrho &= 1200\,\mathrm{kg\,m^{-3}} \\
E_0 &= 8.50 \cdot 10^6\ \mathrm{Pa} \\
E_1 &= 0.51 \cdot 10^6\ \mathrm{Pa} \\
\varkappa &= 0.08 \cdot 10^6\ \mathrm{Pa\,s}
\end{aligned}
$$

$$\alpha = \frac{E_1}{E_0} = 0.060$$

$$\beta = \frac{\varkappa\omega_0}{E_0} = 8.725$$

filled ethylene-propylene-diene-monomer-rubber