A Brief Survey on Non-standard Constraints

Simulation and Optimal Control

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Non-standard Constraints

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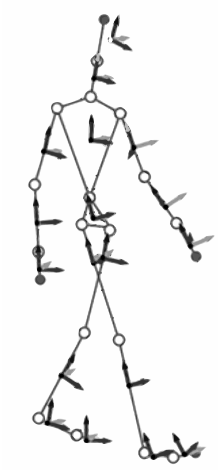

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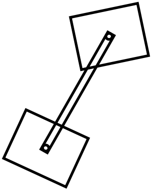
Classification of Constraints

holonomic



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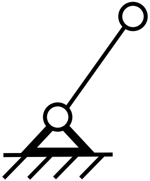
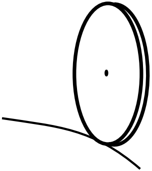


scleronomic

nonholonomic



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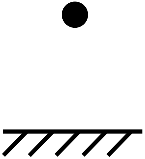
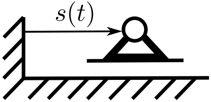


rheonomic



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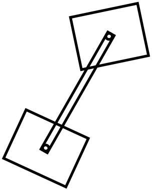

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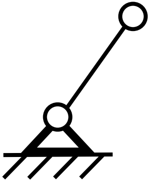

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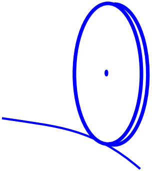

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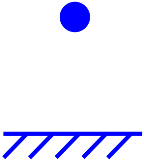

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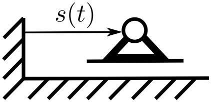


Nonholomic Constraints

Definition

A Holonomic constraint is defined by an equation of the position variables $\boldsymbol{\Psi}(\mathbf{q}) = \mathbf{0}$.

Literally **nonholonomic** means not holonomic, however the common understanding are relations between differentials

$$\Phi(\mathrm{d}\mathbf{q}) = 0$$

which can not be integrated to position level. An obvious sufficient condition for integrability of linear forms $\mathbf{a}(\mathbf{q}) \cdot \mathrm{d}\mathbf{q} = 0$ is

$$\nabla \mathbf{a} = \nabla^{\mathrm{T}} \mathbf{a}.$$

This condition is not necessary, since several constraints may be non-integrable separately, but are integrable taken together.

The Frobenius Theorem[1] gives necessary and sufficient conditions for integrability.

[1] T. Hawkins: Frobenius, Cartan, and the problem of Pfaff (2005)

Formulation

System without constraints are described by ODEs

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{v} \\ \mathbf{M}(\mathbf{q})\dot{\mathbf{v}} &= \mathbf{f}(\mathbf{q}, \mathbf{v}) \end{aligned}$$

Holonomic constraints $\Psi(\mathbf{q}) = 0$ lead to an index-3-DAE

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{v} \\ \mathbf{M}(\mathbf{q})\dot{\mathbf{v}} &= \mathbf{f}(\mathbf{q}, \mathbf{v}) + \left(\frac{\partial \mathbf{\Psi}}{\partial \mathbf{q}}\right)^{\mathrm{T}} \boldsymbol{\lambda} \\ \mathbf{\Psi}(\mathbf{q}) &= \mathbf{0} \end{aligned}$$

Nonholonomic constraints $\mathbf{A}\mathrm{d}\mathbf{q} = 0$ lead to an index-2-DAE

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{v} \\ \mathbf{M}(\mathbf{q})\dot{\mathbf{v}} &= \mathbf{f}(\mathbf{q}, \mathbf{v}) + \mathbf{A}^{\mathrm{T}} \boldsymbol{\lambda} \\ \mathbf{A}(\mathbf{q})\mathbf{v} &= \mathbf{0} \end{aligned}$$

Generic Schemes

Holonomic constraints enter the popular Gear-Gupta-Leimkuhler formulation[2] (stabilized index 2) on both, position and velocity level

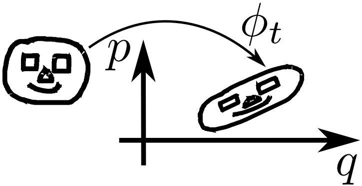
$$
\begin{aligned}
\dot{\mathbf{q}} &= \mathbf{v} - \mathbf{G}^{\mathrm{T}}(\mathbf{q})\boldsymbol{\eta} \\
\mathbf{M}(\mathbf{q})\dot{\mathbf{v}} &= \mathbf{f}(\mathbf{q},\mathbf{v}) - \mathbf{G}^{\mathrm{T}}(\mathbf{q})\boldsymbol{\lambda} - \mathbf{A}^{\mathrm{T}}\boldsymbol{\mu} \\
\boldsymbol{\Psi}(\mathbf{q}) &= \mathbf{0} \\
\mathbf{G}(\mathbf{q})\mathbf{v} &= \mathbf{0} \\
\mathbf{A}(\mathbf{q})\mathbf{v} &= \mathbf{0}
\end{aligned}
$$

Nonholomic constraints are added on velocity level.

- ODE description of unconstrained systems is naturally extended to DAE description when contraints are introduced.
- Nonholonomic constraints in mechanical systems lead to index-2-DAEs and are thus even easier included into generic schemes than holonomic constraints.

[2] C. W. Gear, B. Leimkuhler, and G. K. Gupta: Automatic integration of Euler-Lagrange equations with constraints (1985)

Structure Preservation



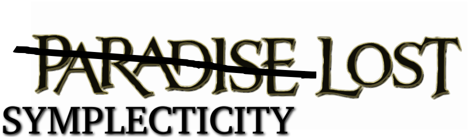
Structure-preserving integrators work very well for holonomic systems.

They preserve first integrals, such as energy and momentum, over long time.

Nonholonomic systems do not generally preserve a symplectic structure.

However, integrators derived from the Discrete D'Alembert Principle perform well on some benchmark problems.

Structure Preservation



D'Alembert's Principle

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\delta \mathbf{q} = \mathbf{0}$$

for **holonomic** systems means

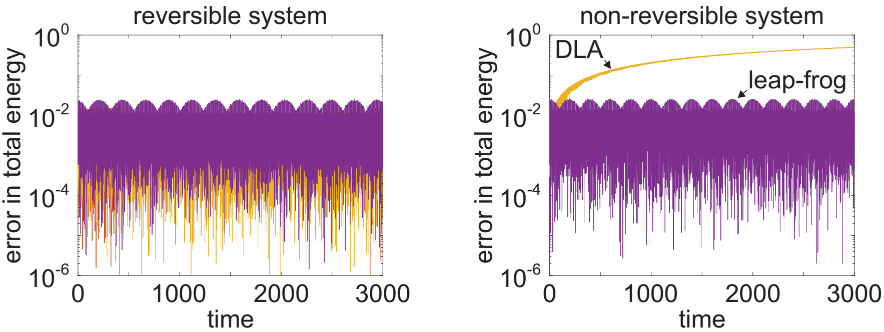
- independent minimal coordinates $\delta\mathbf{q}$ exist,
- corresponds to stationarity of action functional (Hamilton),
- from stationarity of action follows symplecticity (Liouville).

for **nonholonomic** systems means

- $\delta\mathbf{q}$ are not independent,
- the velocity constraints are either enforced by Lagrange-Multipliers (Lagrange Equations of first kind) or by elimination/quasicoordinates,
- anyway, symplecticity is lost.

Structure Preservation

The Discrete D'Alembert Principle is entirely vitiated by altered benchmarks[3].



Well, not entirely... one integrator does suprisingly well.
Leap-Frog still holds out against the altered benchmarks.

[3] K. Modin and O. Verdier: What makes nonholonomic integrators work? (2019)

Structure Preservation

Nonholonomic scleronomic constraints do not affect the energy conservation, consequently energy-consistent schemes[3] are appealing.

Energy conservation is ensured by the **Discrete Gradient**

$$\bar{\nabla} f(\mathbf{q}_n, \mathbf{q}_{n+1}) \cdot (\mathbf{q}_{n+1} - \mathbf{q}_n) = f(\mathbf{q}_{n+1}) - f(\mathbf{q}_n).$$

A direct discretization of

$$\begin{aligned} \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} &= -\nabla V(\mathbf{q}) - \mathbf{G}^{\mathrm{T}}(\mathbf{q})\boldsymbol{\lambda} - \mathbf{A}^{\mathrm{T}}(\mathbf{q})\boldsymbol{\mu} \\ \boldsymbol{\Psi}(\mathbf{q}) &= \mathbf{0} \\ \mathbf{A}(\mathbf{q})\dot{\mathbf{q}} &= \mathbf{0} \end{aligned}$$

is possible, but has two drawbacks, an increased number of unknowns (Lagrange-Multipliers) and conditioning problems.

[3] P. Betsch: A Unified Approach to the Energy-Consistent Numerical Integration of Nonholonomic Mechanical Systems and Flexible Multibody Dynamics (2004)

Structure Preservation

The **discrete null space matrix** $\mathbf{P}$ with the property

$$[\mathbf{G}(\mathbf{q}_n, \mathbf{q}_{n+1})^{\mathrm{T}}, \mathbf{A}(\mathbf{q}_{n+\frac{1}{2}})^{\mathrm{T}}]\, \boldsymbol{P}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \mathbf{0}$$

projects the discretized equations of motions onto the admissible submanifold

$$\begin{aligned} \mathbf{P}^{\mathrm{T}} \left(\frac{2}{\Delta t} \mathbf{M} \left(\mathbf{q}_{n+1} - \mathbf{q}_n \right) - 2\mathbf{M}\mathbf{v}_n + \Delta t \bar{\nabla} V \right) &= \mathbf{0} \\ \boldsymbol{\Psi}(\mathbf{q}_{n+1}) &= \mathbf{0} \\ \mathbf{A}(\mathbf{q}_{n+\frac{1}{2}})(\mathbf{q}_{n+1} - \mathbf{q}_n) &= \mathbf{0}. \end{aligned}$$

Note, the discrete gradient only ensures the conservation of total energy.

Nonholonomic Constraints in Simulations

- ODE description of unconstrained systems is naturally extended to DAE description, when constraints enter.
- In generic schemes nonholonomic constraints are even easier to handle.
- Structure-preservation for nonholonomic systems is non-standard, only energy conservation is evident.
- Symplecticity is lost, so symplectic integrators do not work well in general, but on some benchmark problems.
- Leap-frog performs suprisingly well, demanding further research about the geometric properties of nonholonomic systems.

Optimal Control Problem (feed-forward)

Definition of the Optimal Control Problem (OCP) with nonholonomic constraints

$$
\begin{aligned}
\min_{\mathbf{u},\mathbf{q}} J_C &= \int_{t_b}^{t_e} C\big(\mathbf{q}(t), \dot{\mathbf{q}}, \mathbf{u}(t)\big)\,\mathrm{d}t \\
\text{s.t.: } \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} &= \mathbf{f}\big(\mathbf{q}(t), \dot{\mathbf{q}}(t), \mathbf{u}(t)\big) \\
\mathbf{A}(\mathbf{q})\dot{\mathbf{q}} &= \mathbf{0} \\
\mathbf{q}(t_b) &= \mathbf{q}_0, \qquad \mathbf{q}(t_e) = \mathbf{q}_T \\
\dot{\mathbf{q}}(t_b) &= \dot{\mathbf{q}}_0, \qquad \dot{\mathbf{q}}(t_e) = \dot{\mathbf{q}}_T \\
\mathbf{g}\big(\mathbf{q}(t), \mathbf{u}(t)\big) &\geq 0 \qquad \forall t \in [t_b, t_e].
\end{aligned}
$$

Direct methods appear most promising[3], so we limit the discussion to them.

[3] M. Gerdts: Optimal control of ODEs and DAEs (2011)

Direct Methods

Discretization of both, system dynamics and cost function, leads to a finite-dimensional optimization problem

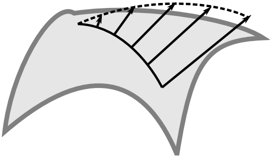
$$\begin{aligned}
\text{min } J_C^d &= \sum_{k=0}^{N-1} C_d(\mathbf{q}_k, \mathbf{q}_{k+1}, \mathbf{u}_k), \\
\text{s.t.: } \mathbf{q}_0 &= \mathbf{q}_s, \qquad \dot{\mathbf{q}}_0 = \dot{\mathbf{q}}_s, \\
\mathbf{0} &= f_d(\mathbf{q}_{k-1}, \mathbf{q}_k, \mathbf{u}_k) \qquad \text{for } k = 1 \dots N-1, \\
\mathbf{q}_N &= \mathbf{q}_e, \qquad \dot{\mathbf{q}}_N = \dot{\mathbf{q}}_e,
\end{aligned}$$

where the nonholonomic constraints may be included into the system dynamics by the discrete null space method, similarly as done for holonomic constraints by DMOCC[4].

[4] S. Leyendecker, S. Ober-Blöbaum, J. E. Marsden, and M. Ortiz: Discrete mechanics and optimal control for constrained systems

Direct Methods

Starting with an possibly infeasible guess $[x, u]$, the functional-based approach[4] goes in 3 steps.



It has been implemented[5] for holonomic constraints using variational integrators.

The extension to nonholonomic constraints is possible with the discrete null space method as well.

[4] J. Hauser: A Projection Operator Approach to the Optimization of Trajectory Functionals (2000)

[5] E.R. Johnson and T.D. Murphey: Scalable variational integrators for constrained mechanical systems in generalized coordinates (2009)

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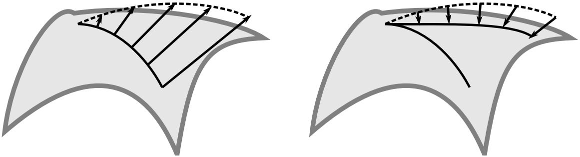

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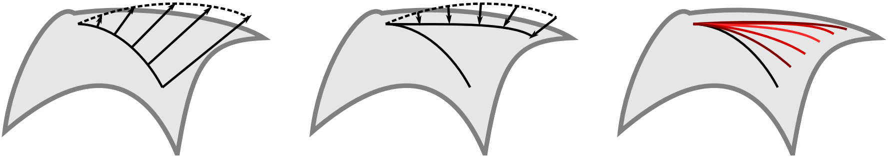

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Nonholonomic Constraints in Optimal Control

Summary

- Direct methods appear more promising than indirect methods (not discussed).
- The discrete null space matrix offers an elegant way to satisfy both, holonomic and nonholonomic, constraints within direct methods.
- Current research aims to identify and utilize further geometric properties of the optimal control problem for nonholonomic systems.

Unilateral Constraints

Definition

In mechanical systems unilateral constraints arise from contacts[6] via

impacts

$$
\begin{aligned}
m\ddot{x} &= f_g + f_c, \\
0 \le x &\perp f_c \ge 0, \\
\dot{x}^+ &= -\varepsilon \dot{x}^- \qquad \text{if} \quad x = 0 \quad \text{and} \quad \dot{x}^- \le 0,
\end{aligned}
$$

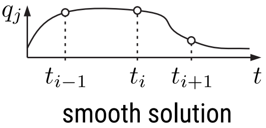
and friction

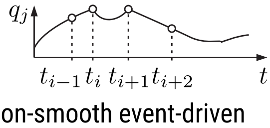
$$
\begin{aligned}
0 &= f_a - f_f & &\text{if } \dot{x} = 0 \text{ and } f_a \le \mu |N|, \\
m\ddot{x} &= f_a - \mu |N| \operatorname{sgn} \dot{x} & &\text{else, with } \operatorname{sgn} 0 = [-1, 1].
\end{aligned}
$$

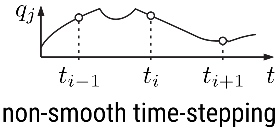
As consequence, solutions may be non-differentiable or even discontinuous.

[6] V. Acary and B. Brogliato: Numerical methods for nonsmooth dynamical systems: applications in mechanics and electronics (2008)

Categorization







Event-driven offers higher accuracy, whereas time-stepping is more robust and and can handle many collisions, even contemporaneously.

Formulation in terms of **complementarity problems**, **variational inequalities** or **differential inclusions** depending on the favored numerical tools (LCP solvers, . . .).

For completeness, regularizations may smooth out discontinuities, but lead to artificially stiff systems.

Event-driven Schemes

A switching functions $s(q,t)$ indicates discontinuities ($s=0$) in the dynamic system

$$\begin{aligned} \dot{q} &= f(q,\lambda,s), \\ 0 &= g(q,\lambda,s). \end{aligned}$$

Event-detection corresponds to root finding[7].

No sign change of $s \rightsquigarrow$ integration goes on.

Sign change $\rightsquigarrow$ detect event, i.e. solve

$$\begin{aligned} g(P^q(\hat{t}),\hat{\lambda}) &= 0 \\ s(P^q(\hat{t}),\hat{\lambda}) &= 0 \end{aligned}$$

for $\hat{t}$ and $\hat{\lambda}$, and restart integration.

[7] L. F. Shampine and S. Thompson: Event location for ordinary differential equations (2000)

Event-driven Schemes

In the context of structure preservation, variational collision integrators[7] can be constructed from Hamilton's principle, when the collision time t_i is variable

$$\delta \int_0^T L(q,\dot{q})\,\mathrm{d}t = \int_0^{t_i} \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} \right) \cdot \delta q\,\mathrm{d}t + \int_{t_i}^T \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} \right) \cdot \delta q\,\mathrm{d}t$$

$$- \left| \frac{\partial L}{\partial \dot{q}} \cdot \delta q + L\delta t \right|_{t_i^-}^{t_i^+}.$$

However, variational inequalities are needed for a full description[8].
The utilization of this modified principle is subject to current research.

[7] R. C. Fetecau, J. E. Marsden, M. Ortiz, and M. West: Nonsmooth Lagrangian mechanics and variational collision integrators (2003)

[8] R. I. Leine, U. Aeberhard, and C. Glocker: Hamilton's principle as variational inequality for mechanical systems with impact (2009)

Time-stepping Schemes

Typically implicit schemes averaging all events over a time step h.

A simple discretization[9] for a simple example (bouncing ball) starts with

$$x_{k+1} = x_k + h\dot{x}_k,$$

depending on the end position, the free flight continues ($x_{k+1} > 0$)

$$m(\dot{x}_{k+1} - \dot{x}_k) = hf_{g,k+1},$$

or the contact is evaluated ($x_{k+1} \leq 0$), here as LCP

$$\begin{aligned} m(\dot{x}_{k+1} - \dot{x}_k) &= hf_{g,k+1} + F_{c,k+1} \\ 0 &\leq \dot{x}_{k+1} + \varepsilon\dot{x}_k \perp F_{c,k+1} \geq 0. \end{aligned}$$

[9] J.J. Moreau: Unilateral contact and dry friction in finite freedom dynamics (1988)

Time-stepping Schemes

In terms of structure preservation the use of variational integrators is evident, since they are formulated in terms of momenta and avoid direct computation of forces.

Adding a non-smooth penalty term to the discrete action enforces hard inequality constraints on the endpoints[9]

$$\delta_k \sum_{k=0}^{N-1} \Big(L_d(q_k, q_{k+1} - I_A(q_{k+1}) \Big) \ni 0.$$

Stationarity then gives us a constrained, Discrete Euler-Lagrange Inclusion

$$\begin{aligned} D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1} - \partial I_A(q_k) \quad &\ni \quad 0, \\ q_{k+1} \quad &\in \quad \mathcal{D}. \end{aligned}$$

[9] D. M. Kaufman and D. K. Pai. Geometric numerical integration of inequality constrained, nonsmooth Hamiltonian systems (2012)

Unilateral Constraints in Simulation

Event-driven

- $-$ not too many events, can not handle infinite events in finite time (Zeno),
- $\pm$ structure preserving methods are on the way,
- $+$ high order methods for smooth parts, highly efficient methods for root finding.

Time-stepping

- $-$ low order methods,
- $\pm$ events are only localized up to time step, constraints slightly violated,
- $+$ robustness, structure preservation is available.

Optimal Control (feed-forward)

$$\min_{u(t),q(t)} J_C = \int_{t_b}^{t_e} C\big(q(t), \dot{q}(t), u(t)\big)\,\mathrm{d}t$$ cost functional

s.t.: $$q(t_b) = q_b, \qquad \dot{q}(t_b) = \dot{q}_b$$ initial conditions

$$q(t_e) = q_e, \qquad \dot{q}(t_e) = \dot{q}_e$$ terminal conditions

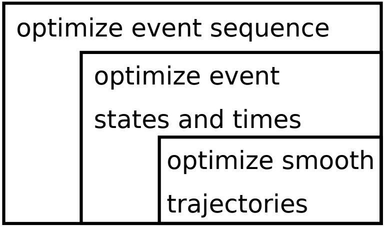
$$\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = f\big(q(t), \dot{q}(t), u(t)\big), \quad q \in \mathcal{D}$$ system dynamics including collisions

$$r\big(q(t), u(t)\big) \geq 0 \qquad \forall t \in [t_b, t_e]$$ control constraints

Note, difference between compliance with unilateral constraints (KKT) and collisions (re-init).

Event-driven Discretization of OCP

An event-driven disretization introduces the collision states and times as additional variables. Consequently, there are three levels of optimization.



The computational efforts blow up and make this approach not seem very promising.

Time-stepping Discretization of OCP

Time-stepping discretizations, a.k.a. contact-implicit formulations[10], avoid separate handling of events.

$$\min_{h,\mathbf{q},\mathbf{u},\mathbf{c}} J(h,\mathbf{q},\mathbf{u}) + \alpha \sum_{k=1}^{N-1} s_k$$

$$\text{s.t.:} \quad \begin{aligned} f(h, q_{k-1}, q_k, q_{k+1}, \lambda_k, \psi_k, \eta_k) &= 0 \\ \mathbf{g}(q_{k+1}, \lambda_k, \psi_k, \eta_k, s_k) &\geq 0 \\ u_{\min} &\leq u_k \leq u_{\max} \\ h_{\min} &\leq h \leq h_{\max} \end{aligned}$$

However, method finds local minima, thus initial guess is decisive.

[10] Z. Manchester, N. Doshi, R. J Wood, and S. Kuindersma. Contact-implicit trajectory optimization using variational integrators (2019)

Unilateral Constraints in Optimal Control

Utilization of collisions and structure switching are on the cutting edge of research.

The optimal control problem is more involved than the simulation, thus simple time discretizations, i.e. time-stepping schemes, are preferred.

Moreover, the computational efforts for event-driven methods to solve the OCP grow exponentially with higher dimensions and collision modes.

Summary

Nonholonomic constraints

- are easily included into the popular Gear-Gupta-Leimkuhler formulation;
- do not, in general, preserve a symplectic structure;
- suggest an energy-momentum conserving framework for long-term simulations and optimal control problems.

Unilateral constraints

- lead to events that are either iteratively detected (*event-driven schemes*) or averaged over a time step (*time-stepping schemes*);
- suggest a reformulation of the variational principles in terms of variational inequalities;
- are favorable included via time-stepping schemes (*contact-implicit*) into the optimal control problem.

Aggregation of holonomic, nonholonomic and unilateral constraints.

Frobenius Theorem

For example, the constraints

$$\begin{aligned} (x^2+y^2)\,\mathrm{d}x + xz\,\mathrm{d}z &= 0, & \nabla \mathbf{g}_1 &= \begin{bmatrix} 2x & 2y & x \\ 0 & 0 & 0 \\ z & 0 & x \end{bmatrix}, \\ (x^2+y^2)\,\mathrm{d}y + yz\,\mathrm{d}z &= 0, & \nabla \mathbf{g}_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 2x & 2y & 0 \\ 0 & z & y \end{bmatrix} \end{aligned}$$

are not integrable separately, but together the can by integrated to

$$\begin{aligned} x^2+y^2+z^2 &= C_1, \\ \frac{x}{y} &= C_2. \end{aligned}$$

Frobenius Theorem

Rolling wheel[11]

Constraints:

$$\begin{aligned}
\pi_1 &= \dot{x}_1 + R\dot{\phi}\cos\psi + R\dot{\psi}\cos\theta\cos\psi - R\dot{\theta}\sin\theta\sin\psi \\
\pi_2 &= \dot{x}_2 + R\dot{\phi}\sin\psi + R\dot{\psi}\cos\theta\sin\psi - R\dot{\theta}\sin\theta\cos\psi \\
\pi_3 &= \dot{x}_3 - R\dot{\theta}\cos\theta
\end{aligned}$$

Coordinates:

$$\mathbf{U} = [x_1, x_2, x_3, \phi, \psi, \theta, t]^{\mathrm{T}}$$

[11] O.M. O'Reilly: Intermediate Dynamics for Engineers (2008)

Frobenius Theorem

differential constraint coefficients

$$\mathbf{W} = \begin{bmatrix} 1 & 0 & 0 & R\cos\psi & R\cos\theta\cos\psi & -R\sin\theta\sin\psi & 0 \\ 0 & 1 & 0 & R\sin\psi & R\cos\theta\sin\psi & R\sin\theta\cos\psi & 0 \\ 0 & 0 & 1 & 0 & 0 & -R\cos\theta & 0 \end{bmatrix}$$

null space of $\mathbf{W}$

$$\begin{aligned} \mathbf{g}_1 &= [0, 0, 0, 0, 0, 0, 1]^{\mathrm{T}} \\ \mathbf{g}_2 &= [0, -R\sin\theta\cos\theta, R\cos^2\theta\cos\psi, 0, \sin\theta\sin\psi, \cos\theta\cos\psi, 0]^{\mathrm{T}} \\ \mathbf{g}_3 &= [-R\sin\theta\cos\theta, 0, -R\cos^2\theta\sin\psi, 0, \sin\theta\cos\psi, -\cos\theta\sin\psi, 0]^{\mathrm{T}} \\ \mathbf{g}_4 &= [R\sin\theta\sin\psi, -R\sin\theta\cos\psi, R\cos\theta, 0, 0, 1, 0]^{\mathrm{T}} \end{aligned}$$

nonzero entries in $S^A_{LK} = W_{AL,K} - W_{AK,L}$

$$\begin{aligned} S^1_{65} = -S^1_{56} = R\sin\psi \quad &\rightsquigarrow \quad \mathbf{g}_2^{\mathrm{T}}\mathbf{S}^1\mathbf{g}_3 \neq 0 \\ S^2_{65} = -S^2_{56} = R\cos\psi \quad &\rightsquigarrow \quad \mathbf{g}_2^{\mathrm{T}}\mathbf{S}^2\mathbf{g}_3 \neq 0 \end{aligned}$$

Frobenius Theorem

reduction to planar rolling $x_2 = 0$ and $\psi = 0$

$$\mathbf{W} = \begin{bmatrix} 1 & 0 & 0 & R & R\cos\theta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & R\sin\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & -R\cos\theta & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

null space of $\mathbf{W}$

$$\begin{aligned} \mathbf{g}_1 &= [0,0,0,0,0,0,1]^{\mathrm{T}} \\ \mathbf{g}_2 &= [-R,0,0,1,0,0,0]^{\mathrm{T}} \end{aligned}$$

all $\mathbf{g}_L^{\mathrm{T}} \mathbf{S}^A \mathbf{g}_K = 0$ for all $L \neq K \rightsquigarrow$ system of constraints is integrable

Experimental Validation

GAMM Student Chapter Chemnitz