A Variational Approach to Optimal Control of Underactuated Mechanical Systems with Collisions

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Motivation

http://www.theregister.co.uk

Discrepancies between real systems and multi-body models:

- **contact (collisions, friction)**
- deformation (elastic, plastic, ...)
- physical domain couplings
- parameter uncertainties
- interactions with the environment

Outline of the Talk and Literature behind

- Non-smooth Dynamics
  - C. Glocker
  - B. Brogliato
- Variational Integrators and Optimal Control
  - J.E. Marsden
  - T. Murphey, S. Leyendecker
  - O. Junge, S. Ober-Blöbaum
- Variational Integrators and Optimal Control for Non-smooth Systems
  - J.E. Marsden, R.C. Fetecau
  - T. Murphey, D. Pekarek
  - S. Ober-Blöbaum, K. Flaßkamp
  - S. Leyendecker, M. Koch
- Pendulum on Cart
  - A. Astrom, K. Furuta
  - ...

Non-smooth Dynamics

Hybrid systems combine time continuous and time discrete dynamics.
In mechanical systems discrete events originate from collision or friction modeling.

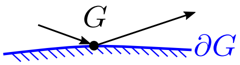
HAMILTON's Principle in generalized coordinates $q \in \mathbb{R}^n$

$$0 = \delta \int_{t_B}^{t_E} \mathcal{L}(q, \dot{q}, t)\,\mathrm{d}t + \int_{t_B}^{t_E} \underbrace{Q(q, \dot{q}, t) \cdot \delta q}_{\delta W^{\mathsf{nc}}}\,\mathrm{d}t$$

subjected to unilateral constraint

$$g(q) \leq 0$$

admissible positions $q \in G$, collisions at boundary ∂G.



Variational Formulation

HAMILTON's principle for a time interval with a collision at time $t = t_i, \quad t_B < t_i < t_E$

$$0 = \delta \int\limits_{t_B}^{t_E} \mathcal{L}\,\mathrm{d}t + \int\limits_{t_B}^{t_E} Q \cdot \delta q\,\mathrm{d}t$$

$$\begin{aligned} 0 = & \int\limits_{t_B}^{t_i^-} \left(\mathcal{L}_q - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\dot{q}} + Q\right) \cdot \delta q\,\mathrm{d}t + \left|\mathcal{L}_{\dot{q}} \cdot \delta q + \mathcal{L}\delta t_i\right|_{t_B}^{t_i^-} \\ & + \int\limits_{t_i^+}^{t_E} \left(\mathcal{L}_q - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\dot{q}} + Q\right) \cdot \delta q\,\mathrm{d}t + \left|\mathcal{L}_{\dot{q}} \cdot \delta q + \mathcal{L}\delta t_i\right|_{t_i^+}^{t_E} + \sigma(t_i)\hat{Q} \cdot \delta q. \end{aligned}$$

Generalized coordinates $q(t)$ are assumed continuous, velocities/momenta not.

Variational Formulation

Impact position is restricted to $g\big(q(t_i)\big) = 0$, impact time t_i depends on $q(t)$

$$\begin{aligned} 0 &\;=\; \delta g\big(q(t_i)\big) = \nabla g \cdot \big(\delta q(t_i) + \dot{q}(t_i)\delta t_i\big) \\ \delta t_i = 0 &\;\rightsquigarrow\; \nabla g \cdot \delta q(t_i) = 0 \rightsquigarrow \delta q(t_i) \in T\partial G \\ \delta t_i \neq 0 &\;\rightsquigarrow\; \delta q(t_i) = -\dot{q}(t_i)\delta t_i \end{aligned}$$

thus boundary term turns for frictionless elastic collisions (smooth Q) into

$$0 = -\left|\mathcal{L}_{\dot{q}} \cdot \delta q + \mathcal{L}\delta t_i\right|_{t_i^-}^{t_i^+} = \begin{cases} -\left|\mathcal{L}_{\dot{q}} \cdot \delta q_{\mathbf{t}}\right|_{t_i^-}^{t_i^+} & \text{for } \delta t_i = 0 \\ -\left|(-\dot{q} \cdot \mathcal{L}_{\dot{q}} + \mathcal{L})\delta t\right|_{t_i^-}^{t_i^+} & \text{otherwise} \end{cases}$$

which represents *conservation of momentum* in tangential directions and *conservation of energy*, respectively.

Variational Integrators (VI)

1.) Approximation of the generalized coordinates per time step $h = t_{k+1} - t_k$

$$q(t) \approx q^d(t) = q_k + \frac{t - t_k}{h}(q_{k+1} - q_k)$$

2.) and numerical integration of the action functional (same for virtual work)

$$\int_{t_k}^{t_{k+1}} \mathcal{L}\big(q(t), \dot{q}(t), t\big)\,\mathrm{d}t \approx h\mathcal{L}\big(q^d(t_{k+1/2}), \dot{q}^d(t_{k+1/2}), t_{k+1/2}\big) = L_d(q_k, q_{k+1}, h)$$

leads to Discrete EULER-LAGRANGE equations (collision-free case)

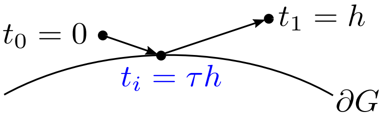
$$\begin{aligned} p_k &= -D_1 L_d(q_k, q_{k+1}, h) - f_d^-(q_k, q_{k+1}, h) \\ p_{k+1} &= D_2 L_d(q_k, q_{k+1}, h) + f_d^+(q_k, q_{k+1}, h). \end{aligned}$$

Notation: $D_1 L_d(q_k, q_{k+1}, h) = \frac{\partial}{\partial q_k} L_d(q_k, q_{k+1}, h)$,

$D_2 L_d(q_k, q_{k+1}, h) = \frac{\partial}{\partial q_{k+1}} L_d(q_k, q_{k+1}, h)$,

$D_3 L_d(q_k, q_{k+1}, h) = \frac{\partial}{\partial h} L_d(q_k, q_{k+1}, h)$.

Collision Time Step



before collision $t_0 \leq t < \tau h \quad \rightsquigarrow \quad q_\tau, \tau$

$$\begin{aligned} \delta q_0 : \; p_0 &= -D_1 L_d(q_0, q_\tau, \tau h) - f_d^-(q_0, q_\tau, \tau h) \\ 0 &= g(q_\tau) \end{aligned}$$

collision at time $t = \tau h \quad \rightsquigarrow \quad q_1$, **remember** $\delta q = \delta q_{\mathbf{t}} - \dot{q}\delta t_i$

$$\begin{aligned} \delta q_\tau : \; 0 = \; & \Big(D_2 L_d(q_0, q_\tau, \tau h) + f_d^+(q_0, q_\tau, \tau h) \\ & + D_1 L_d\big(q_\tau, q_1, (1-\tau)h\big) + f_d^-\big(q_\tau, q_1, (1-\tau)h\big) \Big) \cdot \mathbf{t}_g \\ \delta t_i : \; 0 = \; & D_3 L_d(q_0, q_\tau, \tau h) - f_d^+(q_0, q_\tau, \tau h) \frac{q_\tau - q_0}{\tau h} \\ & - D_3 L_d\big(q_\tau, q_1, (1-\tau)h\big) - f_d^-\big(q_\tau, q_1, (1-\tau)h\big) \frac{q_1 - q_\tau}{(1-\tau)h} \end{aligned}$$

after collision $\tau h < t < t_1 \quad \rightsquigarrow \quad p_1$

$$\delta q_1 : \; p_1 = D_2 L_d\big(q_\tau, q_1, (1-\tau)h\big) + f_d^+\big(q_\tau, q_1, (1-\tau)h\big)$$

Discrete Mechanics and Optimal Control (DMOC)

direct approach, local optimization, originally for smooth trajectories

$$\min_{u(t),\,q(t)} J_C = \int_{t_b}^{t_e} C\big(q(t), \dot{q}(t), u(t)\big)\,\mathrm{d}t$$ cost functional

s.t.:

$$q(t_b) = q_b, \qquad \dot{q}(t_b) = \dot{q}_b$$ initial conditions

$$q(t_e) = q_e, \qquad \dot{q}(t_e) = \dot{q}_e$$ terminal conditions

$$\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = f\big(q(t), \dot{q}(t), u(t)\big)$$ system dynamics including collisions

$$r\big(q(t), u(t)\big) \geq 0 \qquad \forall t \in [t_b, t_e]$$ control constraints

Discrete Mechanics and Optimal Control (DMOC)

DMOC uses the same discretization for the cost functional as for the system dynamics

$$\begin{aligned} C_d(q_k, q_{k+1}, u_k, u_{k+1}) &\approx \int_{t_k}^{t_{k+1}} C\big(q^d(t), \dot{q}^d(t), u^d(t)\big)\,\mathrm{d}t \\ &\approx hC\big(q^d(t_{k+1/2}), \dot{q}^d(t_{k+1/2}), u^d(t_{k+1/2})\big) \end{aligned}$$

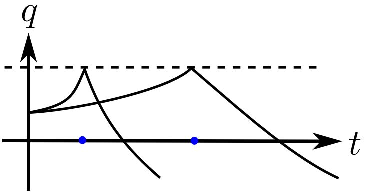
leading to a nonlinear finite dimensional constrained optimization problem..

Discrete Mechanics and Optimal Control (DMOC)

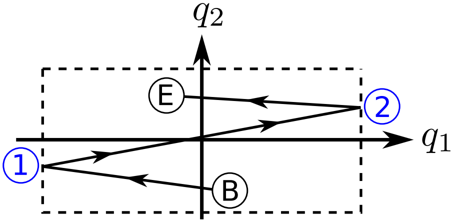
$$\begin{aligned}
\min J_C^d =\; & \sum_{i=0}^{I} \sum_{k=0}^{N_i-1} C_d(q_k^i, q_{k+1}^i, u_k^i, u_{k+1}^i, h_i) && \text{cost function} \\
\text{s.t.: } q_B =\; & q_0^0 && \text{initial positions} \\
p_B =\; & -D_1 L_d(q_0^0, q_1^0) - F_{0,0}^- && \text{initial momenta} \\
q_E =\; & q_N^I && \text{terminal positions} \\
p_E =\; & D_2 L_d(q_{N-1}^I, q_N^I) + F_{I,N-1}^+ && \text{terminal momenta} \\
\text{for } & i = 0 \ldots I \quad \text{and} \quad k = 1 \ldots N_i - 1 \\
0 =\; & D_2 L_d(q_{k-1}^i, q_k^i) + D_1 L_d(q_k^i, q_{k+1}^i) + F_{k-1}^+ + F_k^- && \text{DEL (momentum)} \\
0 =\; & D_3 L_d(q_{N-1}^i, q_N^i, h^i) - F_{k-1}^+ \frac{q_N^i - q_{N-1}^i}{h^i} \\
& -D_3 L_d\left(q_0^{i+1}, q_1^{i+1}, h^{i+1}\right) - F_k^- \frac{q_1^{i+1} - q_0^{i+1}}{h^{i+1}} && \text{collision equations} \\
0 \leq\; & r_d(q_k^i, q_{k+1}^i, u_k^i, u_{k+1}^i) && \text{control constraints}
\end{aligned}$$

..solvable by numerical routines, e.g. SQP, for local extrema.

Incorporating Collisions

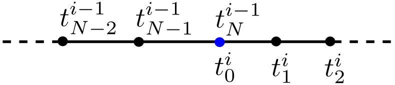


collision velocity/momentum depends on variable interval length ($T_i = N_i h_i$)



collision sequence is defined by its positions (not necessarily all coordinates)

Inner Optimization



$$q_B, p_B, t_B \to q_{c1}, p_{c1}^-, t_{c1} \to q_{c2}, p_{c2}^-, t_{c2} \to \dots \to q_{cN}, p_{cN}^-, t_{cI} \to q_E, p_E$$

The optimization problem for a fixed sequence of I collisions is given by

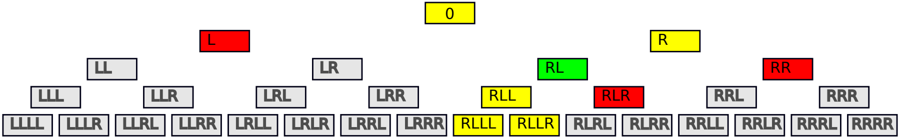
input: initial state $[q_B, p_B]$ at $t = t_B$,
terminal state $[q_E, p_E]$,
collision sequence $[q_{c1}, q_{c2}, \dots, q_{cI}]$;

output: optimal time steps h_i (equidistant per interval)
$\to$ optimal collision momenta p_{ci}^- and optimal collision times t_{ci},
optimal final time t_E,
optimal force $u_i(t)$,
trajectory $q_i(t)$ (verified in simulation),

where $i = 0 \dots I$ is the index for the time intervals of smooth motion.

Outer Optimization

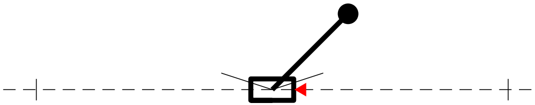
The optimization of the collision sequence is an integer programming problem.
categorization into: optimal, feasible (non-optimal), infeasible, unreachable.



example tree for two possible collisions (left/right)

Computations grow exponentially with number of collisions for exhaustive search, so only the branches around qualified initial guesses can be evaluated.

Pendulum on Cart with Limiters

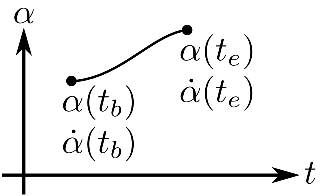


swing-up via one collision

$$\begin{aligned} q &= [\alpha, x]^T \\ \mathcal{L} &= \frac{1}{2}\dot{q}^T M \dot{q} - V(q) \\ M &= \begin{bmatrix} m_p l^2 & -m_p l \sin\alpha \\ \text{sym.} & m_p + m_c \end{bmatrix} \\ V &= m_p l g \sin\alpha \\ \delta W^{\text{nc}} &= F\delta x \end{aligned}$$

$$\begin{aligned} \alpha_{\min} \leq\ & \alpha \leq \alpha_{\max} \quad \text{(hard)} \\ x_{\min} \leq\ & x \leq x_{\max} \quad \text{(soft)} \\ F_{\min} \leq\ & F \leq F_{\max} \end{aligned}$$

implementation issue #1: find initial guess for DMOC

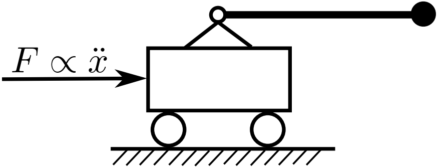


Iterate end time t_e until cart force F does not exceed $F_{\mathrm{min/max}}$:

1. spline approximation $\alpha(t)$ matching I.C. $\alpha(t_b)$, $\dot{\alpha}(t_b)$ and T.C. $\alpha(t_e)$, $\dot{\alpha}(t_e)$;
2. determine $x(t)$ from $\ddot{x}\sin\alpha = l\ddot{\alpha} + g\cos\alpha$ with I.C. $x(t_b)$, $\dot{x}(t_b)$;
3. determine $F(t)$ from $(m_p + m_c)\ddot{x} - m_p l(\ddot{\alpha}\sin\alpha + \dot{\alpha}^2\cos\alpha) = F(t)$.

Fulfillment of $\dot{x}(t_e)$ and $x_{\mathrm{min}} \leq x \leq x_{\mathrm{max}}$ is left for the optimization.

implementation issue #2: local loss of controllability



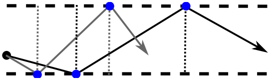
Cart acceleration generates no torque, when pendulum passes the horizontal ($\alpha = k\pi$).

To overcome the local loss of controllability for initial guesses:

- pendulum angle is only prescribed inside controllable region $|\alpha - k\pi| > \Delta\alpha$,
- and system left to its free dynamics ($\ddot{x} = 0$) outside.

implementation issue #3: feasible collision times

Whereas actuator saturation enters the optimization as box-constraints, the minimal collision times (and thus time steps) are interdependent and thus passed as nonlinear inequality constraints.



To estimate the minimal feasible collision times and provide an evaluable gradient thereof, the equations of motion with full force ($F = F_{\min/\max} = \text{const.}$) are simplified using average values (denoted by overbar)

$$\ddot{x} = \frac{F}{m_p + m_c} + \frac{m_p l}{m_p + m_c}(\ddot{a}\sin\alpha + \dot{\alpha}^2\cos\alpha) \approx \frac{F}{m_p + m_c} = \ddot{\bar{x}}$$

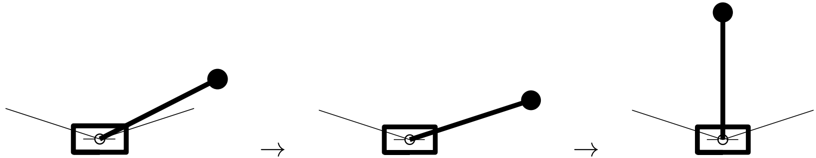
$$\ddot{\alpha} = \frac{1}{l}(\ddot{x}\sin\alpha - g\cos\alpha) \approx \frac{\ddot{\bar{x}}\overline{\sin\alpha}}{l} - \frac{g\overline{\cos\alpha}}{l} = \ddot{\bar{\alpha}}$$

$$\rightsquigarrow \quad \alpha(t) \approx \frac{1}{2}\ddot{\bar{\alpha}}t^2 + \dot{\alpha}_0 t + \alpha_0 \quad \rightsquigarrow T_{\min} \quad \rightsquigarrow h_{\min} = \frac{T_{\min}}{N}$$

with $\dot{\alpha}_0$ and α_0 either from the initial conditions or the collision equations.

Results

optimal trajectory in the sense of $J_C = \int\limits_{t_B}^{t_E} \frac{1}{2} F(t)^2 \, \mathrm{d}t$



$\alpha = \alpha_{\min} + \frac{\pi}{20}, \mathbf{p} = \mathbf{0}$ $\rightarrow$ collision ($\alpha = \alpha_{\min}$) $\rightarrow$ $\alpha = \frac{1}{2}\pi, \mathbf{p} = \mathbf{0}$

0

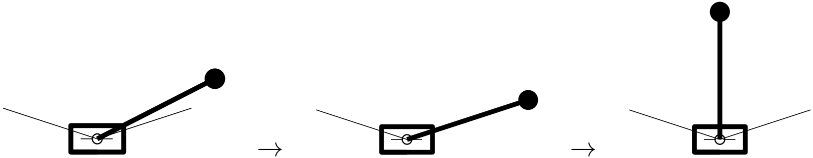
L R*

LL LR RL RR

LLL LLR LRL LRR RLL RLR RRL RRR

Results

optimal trajectory in the sense of $$J_C = \int_{t_B}^{t_E} \tfrac{1}{2} F(t)^2 \, \mathrm{d}t$$



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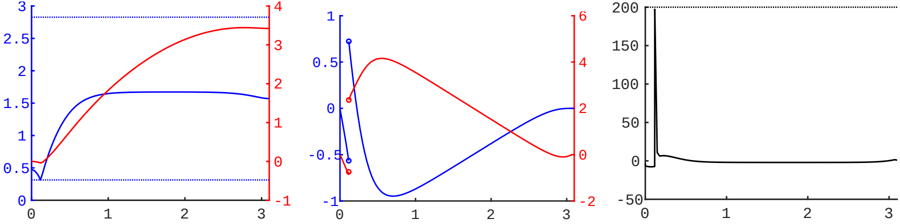

positions $\alpha(t)$, $x(t)$ — momenta $p_\alpha(t)$, $p_x(t)$ — cart force $F(t)$

Summary

- Variational formulation enables construction of Variational Integrators (VI) and Direct Optimal Control (DMOC).
- Introduction of time steps as optimization variables allows non-smoothness of momenta (collisions) during motion.
- Works for nonlinear systems with few degrees of freedom and few collisions.
- Finding initial guesses and limits of feasibility is as difficult as optimization itself.

Limitations

- Only local minima found (inner as well as outer optimization).
- Bounce-back is useful for *energy-pumping*, but there is no limit (*Zeno*).
- There is no other indicator for infeasible tasks, than optimization takes *too long*.

Parameter Values of Example

m_p	$1\,\mathrm{kg}$
m_c	$1\,\mathrm{kg}$
l	$0.5\,\mathrm{m}$
g	$10\,\mathrm{ms}^{-2}$
$\alpha_{\min}$	$\frac{1}{10}\pi$
$\alpha_{\max}$	$\frac{9}{10}\pi$
$x_{\min}$	$-10\,\mathrm{m}$
$x_{\max}$	$10\,\mathrm{m}$
$u_{\min}$	$-10(m_p + m_c)g = -200\,\mathrm{N}$
$u_{\max}$	$10(m_p + m_c)g = +200\,\mathrm{N}$

Parameter Values of Simulation

$N_0 = N_1$	100
TOL	$1e - 14$