Energy-optimal Swing-up of an Electro-mechanically Actuated Pendulum

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Motivation for using Variational Integrators

general advantages

- robustness and excellent long-time behavior
- symplecticity
- backward error analysis

advantages for control

- VI lead to one step maps, which may be formulated as discrete state space systems, for which many existing optimal control methods apply
- VI based controllers require only position but no velocity data

Outline

Foundation

HAMILTON'S PRINCIPLE rules the classical mechanics

$$\delta \int_{t_b}^{t_e} \mathcal{L}\,\mathrm{d}t = 0 \qquad \text{with} \qquad \mathcal{L} = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}),$$

typically used for equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0,$$

which are often nonlinear and need to be solved numerically.

Conservative Systems

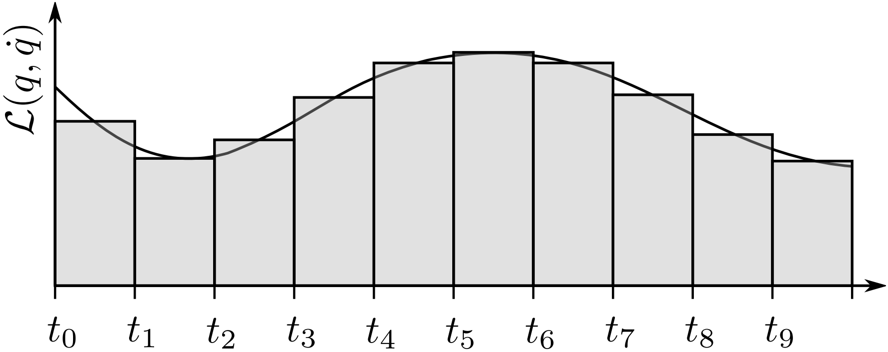
1. Approximation of the state variables in time

$$\mathbf{q}(t) \approx \mathbf{q}^d(t) = M_1(t)\mathbf{q}_k + M_2(t)\mathbf{q}_{k+1}.$$

2. Time-step-wise quadrature of the action-integral

$$\begin{aligned}\Delta S &= \int_{t_k}^{t_{k+1}} \mathcal{L}\big(\mathbf{q}(t), \dot{\mathbf{q}}(t), t\big)\,\mathrm{d}t \\ &\approx \int_{t_k}^{t_{k+1}} \mathcal{L}\big(\mathbf{q}^d(t), \dot{\mathbf{q}}^d(t), t\big)\,\mathrm{d}t \\ &\approx h\mathcal{L}\big(\mathbf{q}^d(t_{k+1/2}), \dot{\mathbf{q}}^d(t_{k+1/2}), t_{k+1/2}\big) = L_d.\end{aligned}$$

Conservative Systems



numerical integration of the Lagrangian by the forward-rectangle rule

Forcing and Dissipation

Same for virtual work of the dissipative and external forces

$$\begin{aligned} \delta W^{\text{nc}} &= \int_{t_k}^{t_{k+1}} \mathbf{F} \cdot \delta \mathbf{q} \, \mathrm{d}t \approx \int_{t_k}^{t_{k+1}} \mathbf{F} \cdot \delta \mathbf{q}^d \, \mathrm{d}t \\ &\approx h \mathbf{F}(t_k) \cdot \delta \mathbf{q}^d(t_k) = \mathbf{F}_k^- \delta \mathbf{q}_k + \mathbf{F}_k^+ \delta \mathbf{q}_{k+1}. \end{aligned}$$

Discrete equations of motion (position-momentum form)

$$\begin{aligned} \mathbf{p}_k &= -D_1 L_d(\mathbf{q}_k, \mathbf{q}_{k+1}) - \mathbf{F}_k^-(\mathbf{q}_k, \mathbf{q}_{k+1}, \mathbf{u}_k) \\ \mathbf{p}_{k+1} &= D_2 L_d(\mathbf{q}_k, \mathbf{q}_{k+1}) + \mathbf{F}_k^+(\mathbf{q}_k, \mathbf{q}_{k+1}, \mathbf{u}_k). \end{aligned}$$

D_i denotes derivative with respect to the ith argument,

i.e. $D_1 L_d(\mathbf{q}_k, \mathbf{q}_{k+1}) = \frac{\partial L_d}{\partial \mathbf{q}_k}$, $D_2 L_d(\mathbf{q}_k, \mathbf{q}_{k+1}) = \frac{\partial L_d}{\partial \mathbf{q}_{k+1}}$.

Electromechanical Systems

Hamilton's principle in charge-displacement formulation

$$\delta \int_{t_b}^{t_e} (T^* + W_m^* - V - W_e)\,\mathrm{d}t + \int_{t_b}^{t_e} \delta W^{\mathsf{nc}}\,\mathrm{d}t = 0$$

symbol	name	example
T^*	kinetic coenergy	$\frac{1}{2}m\dot{x}^2$
W_m^*	magnetic coenergy	$\frac{1}{2}L\dot{q}^2$
V	potential energy	$\frac{1}{2}kx^2$
W_e	electrical energy	$\frac{1}{2}\frac{1}{C}q^2$
δW_m^{nc}	mechanical virtual work	$-d\dot{x}\delta x$
δW_e^{nc}	electrical virtual work	$-R\dot{q}\delta q$

Optimal Control

approaches to optimal control

- discretization-based
  - **discrete Mechanics and optimal control (DMOC)**
  - other integrators, other orders of approximation
    basic problem is dimension of constrained optimization
- functional-based
  - maximum principle
  - dynamic programming
  - Banach-space methods using local methods
    basic problems are inequality constraints

Feed-forward

Feed-forward control determines a nominal trajectory that minimizes a cost functional constrained by the dynamics

$$
\begin{aligned}
\min_{\mathbf{u}} J_C &= \int_{t_b}^{t_e} C\big(\mathbf{q}(t), \dot{\mathbf{q}}, \mathbf{u}(t)\big)\, \mathrm{d}t \\
\text{wrt: } \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} &= \mathbf{f}\big(\mathbf{q}(t), \dot{\mathbf{q}}(t), \mathbf{u}(t)\big) \\
\mathbf{q}(t_b) &= \mathbf{q}_0, \qquad \mathbf{q}(t_e) = \mathbf{q}_T \\
\dot{\mathbf{q}}(t_b) &= \dot{\mathbf{q}}_0, \qquad \dot{\mathbf{q}}(t_e) = \dot{\mathbf{q}}_T \\
\mathbf{g}\big(\mathbf{q}(t), \mathbf{u}(t)\big) &\geq 0 \qquad \forall t \in [t_b, t_e].
\end{aligned}
$$

Feed-forward

DMOC uses the same discretization for the cost functional as for the system dynamics

$$
\begin{aligned}
C_d(\mathbf{q}_k, \mathbf{q}_{k+1}, \mathbf{u}_k) &\approx \int_{t_k}^{t_{k+1}} C\big(\mathbf{q}^d(t), \dot{\mathbf{q}}^d(t), \mathbf{u}^d(t)\big)\,\mathrm{d}t \\
&\approx hC\big(\mathbf{q}^d(t_{k+1/2}), \dot{\mathbf{q}}^d(t_{k+1/2}), \mathbf{u}^d(t_{k+1/2})\big)
\end{aligned}
$$

leading to a nonlinear finite dimensional constrained optimization problem...

Feed-forward

$$\begin{aligned}
\text{min } J_C^d &= \sum_{k=0}^{N-1} C_d(\mathbf{q}_k, \mathbf{q}_{k+1}, \mathbf{u}_k) \\
\text{wrt: } \mathbf{q}_0 &= \mathbf{q}_s \\
\mathbf{0} &= D_2 L(\mathbf{q}_s, \dot{\mathbf{q}}_s) + D_1 L_d(\mathbf{q}_0, \mathbf{q}_1) + \mathbf{F}_0^- \\
\\
k &= 1 \ldots N-1 \\
\mathbf{0} &= D_2 L_d(\mathbf{q}_{k-1}, \mathbf{q}_k) + D_1 L_d(\mathbf{q}_k, \mathbf{q}_{k+1}) + \mathbf{F}_{k-1}^+ + \mathbf{F}_k^- \\
\mathbf{0} &\leq h_d(\mathbf{q}_k, \mathbf{q}_{k+1}, \mathbf{u}_k) \\
\\
\mathbf{q}_N &= \mathbf{q}_e \\
\mathbf{0} &= D_2 L_d(\mathbf{q}_{N-1}, \mathbf{q}_N) - D_2 L(\mathbf{q}_e, \dot{\mathbf{q}}_e) + \mathbf{F}_{N-1}^+
\end{aligned}$$

solvable by numerical routines (e.g. SQP) for local extrema.

Feed-back

Feed-back control stabilizes this nominal trajectory by LQR-control of local linearizations around it

$$\min_{\Delta u} J_L = \frac{1}{2}\int_{t_b}^{t_e} \Delta\mathbf{x}^T(t)\mathbf{Q}(t)\Delta\mathbf{x}(t) + \Delta\mathbf{u}^T(t)\mathbf{R}(t)\Delta\mathbf{u}(t)\,\mathrm{d}t$$

$$\text{wrt: } \Delta\dot{\mathbf{x}} = \begin{bmatrix} \Delta\dot{\mathbf{q}} \\ \Delta\dot{\mathbf{p}} \end{bmatrix} = \mathbf{A}(t)\Delta\mathbf{x} + \mathbf{B}(t)\Delta\mathbf{u}.$$

Feed-back

In discretized form (omitting Δ from now on)

$$\begin{aligned}
\min_{\mathbf{u}} J_L^d &= \sum_{k=0}^{N-1} \Big(\mathbf{x}^T(k)\mathbf{Q}(k)\mathbf{x}(k) + \mathbf{u}(k)\mathbf{R}(k)\mathbf{u}(k) \Big) \\
&\quad + \mathbf{x}^T(N)\mathbf{Q}(N)\mathbf{x}(N) \\
\text{wrt: } \mathbf{x}(0) &= \mathbf{x}_0 \\
\mathbf{x}(k+1) &= \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k).
\end{aligned}$$

This is a standard problem as soon as the linearizations, i.e. system matrix $\mathbf{A}(k)$ and input matrix $\mathbf{B}(k)$, are found. However, such a description restricts the quadrature rules to be causal with respect to the input

$$\mathbf{F}_k^{\pm} = \mathbf{F}^{\pm}(\mathbf{q}_{k-1}, \mathbf{q}_k, \mathbf{u}_{k-1}) \neq \mathbf{F}^{\pm}(\mathbf{q}_{k-1}, \mathbf{q}_k, \mathbf{u}_{k-1}, \mathbf{u}_k).$$

Feed-back

The forced discrete Euler-Lagrange-equations (DEL)

$$\begin{aligned} \mathbf{0} &= \mathbf{p}_k + D_1 L_d(\mathbf{q}_k, \mathbf{q}_{k+1}) + \mathbf{F}_k^-(\mathbf{q}_k, \mathbf{q}_{k+1}, \mathbf{u}_k) \\ \mathbf{p}_{k+1} &= D_2 L_d(\mathbf{q}_k, \mathbf{q}_{k+1}) + \mathbf{F}_k^+(\mathbf{q}_k, \mathbf{q}_{k+1}, \mathbf{u}_k) \end{aligned}$$

implicitly define a one-step map

$$\begin{bmatrix} \mathbf{q}_{k+1} \\ \mathbf{p}_{k+1} \end{bmatrix} = \mathbf{f}_d(\mathbf{q}_k, \mathbf{p}_k, \mathbf{u}_k),$$

whose linearization

$$\begin{bmatrix} \delta\mathbf{q}_{k+1} \\ \delta\mathbf{p}_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{q}_{k+1}}{\partial \mathbf{q}_k} & \frac{\partial \mathbf{q}_{k+1}}{\partial \mathbf{p}_k} \\ \frac{\partial \mathbf{p}_{k+1}}{\partial \mathbf{q}_k} & \frac{\partial \mathbf{p}_{k+1}}{\partial \mathbf{p}_k} \end{bmatrix} \begin{bmatrix} \delta\mathbf{q}_k \\ \delta\mathbf{p}_k \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathbf{q}_{k+1}}{\partial \mathbf{u}_k} \\ \frac{\partial \mathbf{p}_{k+1}}{\partial \mathbf{u}_k} \end{bmatrix} \delta\mathbf{u}_k$$

can be found in explicit expressions.

Feed-back

By differentation of the forced DEL one obtains with $\mathbf{M}_{k+1} = D_2 D_1 L_{k+1} + D_2 \mathbf{F}^-_{k+1}$

$$
\begin{aligned}
\frac{\partial \mathbf{q}_{k+1}}{\partial \mathbf{q}_k} &= -\mathbf{M}^{-1}_{k+1}(D_1 D_1 L_{k+1} + D_1 \mathbf{F}^-_{k+1}) \\
\frac{\partial \mathbf{q}_{k+1}}{\partial \mathbf{p}_k} &= -\mathbf{M}^{-1}_{k+1} \\
\frac{\partial \mathbf{p}_{k+1}}{\partial \mathbf{q}_k} &= (D_2 D_2 L_{k+1} + D_2 \mathbf{F}^+_{k+1}) \frac{\partial \mathbf{q}_{k+1}}{\partial \mathbf{q}_k} \\
&\quad + D_1 D_2 L_{k+1} + D_1 \mathbf{F}^+_{k+1} \\
\frac{\partial \mathbf{p}_{k+1}}{\partial \mathbf{p}_k} &= (D_2 D_2 L_{k+1}) \frac{\partial \mathbf{q}_{k+1}}{\partial \mathbf{p}_k} \\
\frac{\partial \mathbf{q}_{k+1}}{\partial \mathbf{u}_k} &= -\mathbf{M}^{-1}_{k+1} D_3 \mathbf{F}^-_{k+1} \\
\frac{\partial \mathbf{p}_{k+1}}{\partial \mathbf{u}_k} &= (D_2 D_2 L_{k+1} + D_2 \mathbf{F}^+_{k+1}) \frac{\partial q_{k+1}}{\partial u_k} + D_3 \mathbf{F}^+_{k+1}.
\end{aligned}
$$

Feed-back

Now the controller gain

$$\mathbf{K}(k) = (\mathbf{R}_k + \mathbf{B}_k^T \mathbf{P}_{k+1} \mathbf{B}_k)^{-1} \mathbf{B}_k \mathbf{P}_{k+1} \mathbf{A}_k$$

for the feedback law

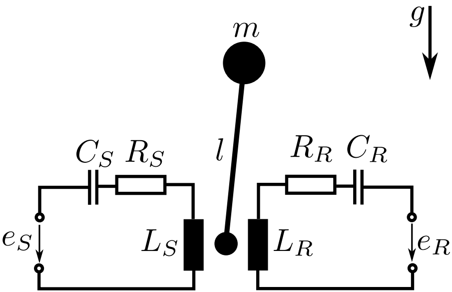
$$\mathbf{u}(k) = \mathbf{u}_n(k) - \mathbf{K}(k)\Big(\mathbf{x}(k) - \mathbf{x}_n(k)\Big)$$

is determined by evaluating the discrete Ricatti equation

$$\begin{aligned}\mathbf{P}_k &= \mathbf{Q}_k + \mathbf{A}_k^T \mathbf{P}_{k+1} \mathbf{A}_k \\ &\quad - \mathbf{A}_k \mathbf{P}_{k+1} \mathbf{B}_k^T (\mathbf{R}_k + \mathbf{B}^T \mathbf{P}_{k+1} \mathbf{B}_k)^{-1} \mathbf{B}_k^T \mathbf{P}_{k+1} \mathbf{A}_k\end{aligned}$$

backward in time from $\mathbf{P}_N = \mathbf{Q}_N$ on.

Electro-Mechanically Actuated Pendulum



Simulation and control by linear approximations in time, mid-point quadrature for the Lagrangian and forward-rectangle quadrature for the virtual work.

Electro-Mechanically Actuated Pendulum

Hamilton's principle in charge-displacement formulation

$$\delta \int_{t_b}^{t_e} (T^* + W_m^* - V - W_e)\,\mathrm{d}t + \int_{t_b}^{t_e} \delta W^{\mathrm{nc}}\,\mathrm{d}t = 0,$$

with

$$\begin{aligned}
T^* &= \frac{ml^2}{2}\dot{\varphi}^2 \\
W_m^* &= \frac{L_R}{2}\dot{q}_R^2 + L_{RS}\cos\varphi\,\dot{q}_R\dot{q}_S + \frac{L_S}{2}\dot{q}_S^2 \\
W_e &= \frac{1}{2C_R}q_R^2 + \frac{1}{2C_S}q_S^2 \\
V &= mgl\sin\varphi \\
\delta W^{\mathrm{nc}} &= e_R\,\delta q_R + e_S\,\delta q_S - R_R\dot{q}_R\,\delta q_R - R_S\dot{q}_S\,\delta q_S.
\end{aligned}$$

Electro-Mechanically Actuated Pendulum

VIs skip the formulation of differential equations, however the equations of motion are valueable for qualified initial guesses in order to get the numerical optimization started

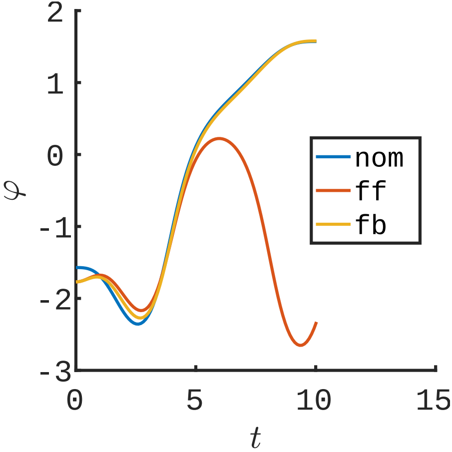
$$\begin{aligned} L_R\ddot{q}_R + R_R\dot{q}_R + \frac{1}{C_R}q_R &= e_R - L_{RS}(\ddot{q}_S\cos\varphi - \dot{q}_S\dot{\varphi}\sin\varphi) \\ L_S\ddot{q}_S + R_S\dot{q}_S + \frac{1}{C_S}q_S &= e_S - L_{RS}(\ddot{q}_R\cos\varphi - \dot{q}_R\dot{\varphi}\sin\varphi) \\ ml^2\ddot{\varphi} + mgl\cos\varphi &= -L_{RS}\sin\varphi\,\dot{q}_R\dot{q}_S. \end{aligned}$$

Initial conditions $\varphi(t_b) = -\frac{\pi}{2}$ mechanically and electrically at rest,
terminal conditions $\varphi(t_e) = \frac{\pi}{2}$ again at rest.

To be found is the input e_R which minimizes the cost functional $J = \int\limits_{t_b}^{t_e} e_R^2\,\mathrm{d}t$, while stator voltage e_S is assumed constant.

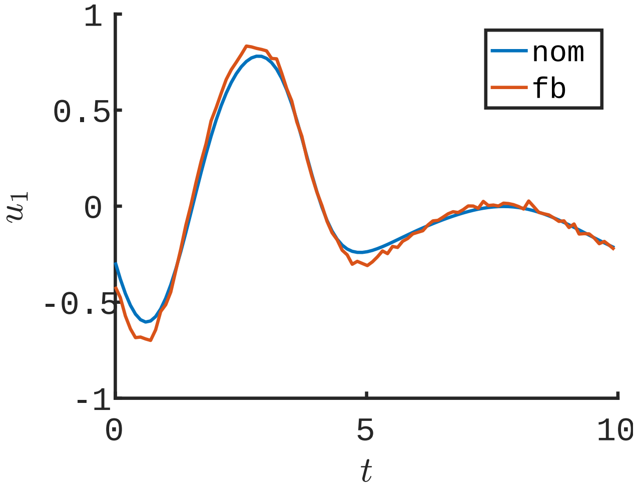
Starting numerical optimization with guess $\varphi_g(t) = -\frac{\pi}{2}\cos\left(\pi\frac{t-t_b}{t_e-t_b}\right)$.

Results



trajectories (pendulum angle): nominal, feed-forward controlled and feedback-controlled in presence of disturbances

Results



control input (rotor circuit voltage): nominal and feedback-controlled in presence of disturbances

Summary

- Hamilton's principle (charge-displacement form) directly discretized (VI),
- optimal control for feed-forward turned into a finite dimensional optimization problem (DMOC),
- structured linearization around the nominal trajectory enabled LQR feed-back control.

Outlook

- the *"small picture"*: improve initial guesses; include second pendulum.
- the *"big picture"*: focus on underactuated systems, hybrid systems; apply DMOC to large-scale models.